**Final - Differential Equations (2018-19)**

*Time: 3 hours. Attempt all questions.*

1. Solve for $u(x, y)$ in the PDE $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$. **[5 marks]**

2. Let $u = u(x, y)$ be a harmonic function in the disk $D = \{r < 1\}$ with $u = \cos\theta + 1$ for $r = 1$.
   (a) Find the maximum value of $u$ in $\bar{D}$. **[2 marks]**
   (b) Calculate the value of $u$ at the origin. **[3 marks]**

3. Solve the heat equation $u_t = ku_{xx}$ for $-l \le x \le l,\ t > 0$, with periodic boundary conditions [i.e. $u(-l, t) = u(l, t)$ and $u_x(-l, t) = u_x(l, t)$ ], and initial profile $u(x, 0) = \phi(x),\ -l \le x \le l$. Assume that $\phi$ is a $C^2$ (twice continuously differentiable) function. **[5 marks]**

4. Consider the wave equation $\rho u_{tt} = Tu_{xx}$ on $-\infty < x < \infty$ with initial profile $\phi(x)$ and initial velocity $\psi(x)$. Assume that $\phi$ and $\psi$ vanish outside the interval $\{|x| \le R\}$. Consider the total energy
$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + Tu_x^2)\, dx.$$
Show that $dE/dt = 0$. (Assume that we can take derivatives inside the integral.) **[5 marks]**

5. Let $\phi(x)$ be any $C^2$ function defined on $\mathbf{R}^3$ that vanishes outside some bounded domain $D$. Show that
$$\phi(0) = -\iiint_D \frac{1}{|x|}\Delta\phi(x)\frac{dx}{4\pi}. \quad \textbf{[6 marks]}$$

6. Consider the Sturm-Liouville equation
$$\frac{d}{dt}\left[p(t)\frac{du}{dt}\right] + [\lambda r(t) - q(t)]u = 0$$
on a closed interval $[a, b]$, with $p, r > 0$ on $[a, b]$ and $p, q, r$ continuously differentiable. Let the endpoint conditions $\alpha u(a) + \alpha' u'(a) = 0,\quad \beta u(b) + \beta' u'(b) = 0$ hold with $\alpha, \alpha', \beta, \beta'$ all being nonzero. Let $u_1$ and $u_2$ be eigenfunctions corresponding to *different* eigenvalues $\lambda_1$ and $\lambda_2$.
   (a) Let $L = D[p(t)D] - q(t)$, where $D$ is the derivative operator. Show that
$$u_1 L[u_2] - u_2 L[u_1] = \frac{d}{dt}\left\{p(t)\big[u_1(t)u_2'(t) - u_2(t)u_1'(t)\big]\right\} \quad \textbf{[4 marks]}$$
   (b) Show that
$$\int_a^b r(t)u_1(t)u_2(t)\, dt = 0. \quad \textbf{[4 marks]}$$

7. Let $\phi : [0, \infty) \to \mathbf{R}$ be a continuous nondecreasing function with $\phi(0) = 0$ and $\phi(x) > 0$ for $x > 0$. Assume further that $\int_\epsilon^1 dx/\phi(x) \to \infty$ as $\epsilon \downarrow 0$.
   (a) Let $a > 0$ and $g : [0, a] \to \mathbf{R}_+$ be a nonnegative continuous function. Let
$$g(x) \le \int_0^x \phi\big(g(t)\big)\, dt, \text{ for } 0 < x \le a.$$
      i. Show that $G(x) = \max_{0 \le t \le x} g(t)$ satisfies the same inequality as $g$. **[4 marks]**
      ii. Let $\bar{G}(x) = \int_0^x \phi\big(G(t)\big)\, dt$. Show that $\bar{G}'(x) \le \phi\big(\bar{G}(x)\big)$. **[4 marks]**
      iii. Conclude that $g(x) \equiv 0$ on $[0, a]$. **[4 marks]**
   (b) Let $F : \mathbf{R} \to \mathbf{R}$ be such that $|F(x) - F(y)| \le \phi(|x - y|)$. Show uniqueness of solutions to the differential equation $\frac{du}{dt} = F(u)$ with $u(0) = u_0 \in \mathbf{R}$. **[4 marks]**